Section 1.9 The matrix of a linear transformation

Example 1. The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that $T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

ANS: Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 \in \mathbb{R}^2$.

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) \overset{\text{property (ii) § 1.8}}{=} x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

$\Rightarrow$ So the formula is $T(\vec{x}) = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix} \left(= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$

$$\begin{array}{cc} \uparrow & \uparrow \\ T(\vec{e}_1) & T(\vec{e}_2) \end{array}$$

Note: We can define a matrix $A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$,

then $T(\vec{x}) = [T(\vec{e}_1) \quad T(\vec{e}_2)]\,\vec{x} = A\vec{x}$

In general, we have:

Theorem 10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose j th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j th column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \tag{1}$$

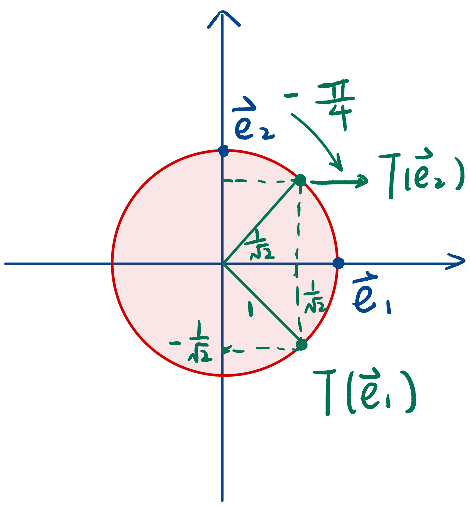
The matrix A in (1) is called the **standard matrix for the linear transformation** T.

Geometric Linear Transformations of $\mathbb{R}^2$ (Check Table 1-4 for more examples)

Example 2. Assume that T is a linear transformation. Find the standard matrix of T.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $-\pi/4$ radians (since the number is negative, the actual rotation is clockwise) [Hint: $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2})$]

We need to find $A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$

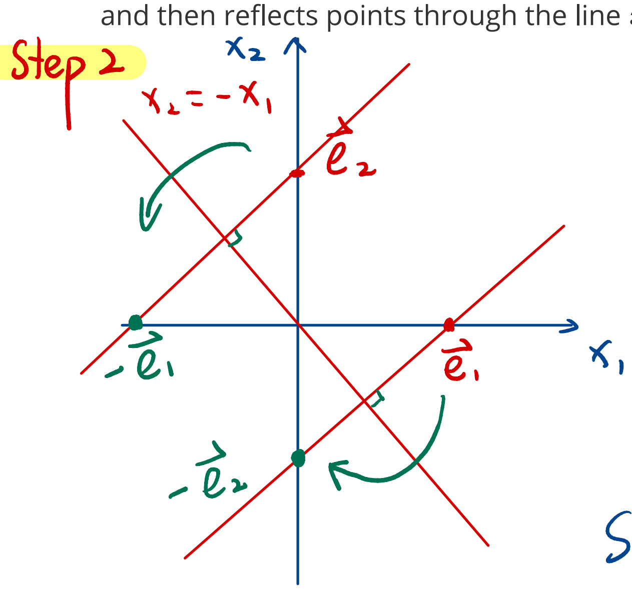


$T(\vec{e}_1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$T(\vec{e}_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(2) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 - 3\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.



ANS: We know under the reflection through the line $x_2 = -x_1$ (step 2)

$\vec{e}_1$ is mapped to $-\vec{e}_2$

$\vec{e}_2$ is mapped to $-\vec{e}_1$

So T maps:

$\vec{e}_1 \xrightarrow{\text{step 1}} \vec{e}_1 \xrightarrow{\text{step 2}} -\vec{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

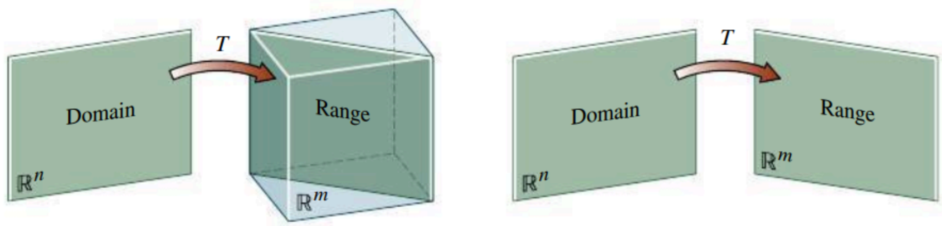
$\vec{e}_2 \xrightarrow{\text{step 1}} \vec{e}_2 - 3\vec{e}_1 \xrightarrow{\text{step 2}} -\vec{e}_1 + 3\vec{e}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Thus $A = [T(\vec{e}_1) \quad T(\vec{e}_2)] = \begin{bmatrix} 0 & -1 \\ -1 & 3 \end{bmatrix}$

Existence and Uniqueness Questions

Definitions

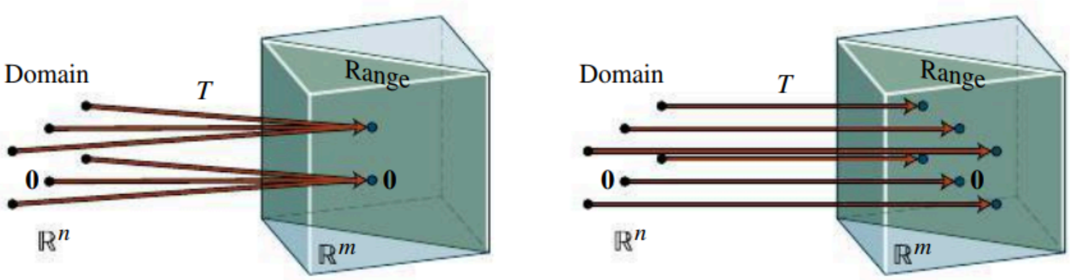
1. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$. This is an existence question.



T is *not* onto $\mathbb{R}^m$ T is onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

2. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$. This is a uniqueness question.



T is *not* one-to-one T is one-to-one

FIGURE 4 Is every $\mathbf{b}$ the image of at most one vector?

Theorem 11. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

proof on page 81

Theorem 12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:
a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$; (Thm 4 in § 1.4)
b. T is one-to-one if and only if the columns of A are linearly independent. $\Leftrightarrow$ $A\vec{x}=\vec{0}$ has only trivial solution

§ 1.7 $\Leftrightarrow$ columns of A are linearly independent

proof on Page 82.

Example 3. Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

ANS: $T(x_1, x_2) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The two columns of A are linearly independent since they are not multiples. Thus by Thm 12 b), T is one-to-one.
Since A is 3×2, the columns of A spans $\mathbb{R}^3$ if and only if A has 3 pivot positions (by Thm 4). As A only has 2 columns, this is impossible! So T is not onto.

Example 4. Describe the possible echelon forms of the standard matrix for the given linear transformation T. Use the notation of Example 1 in Section 1.2.

$T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

ANS: By Thm 12, the columns of the standard matrix A must be linearly independent, and hence the equation $A\vec{x}=\vec{0}$ has no free varibles. So each column of A must be a pivot position.

$$A \sim \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

Note T cannot be onto because of the shape of A (same reason with example 3)

The following two questions are left as exercises. I will provide the complete notes for solving them after the lecture.

Exercise 5. Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$$

ANS: By inspection

$$\begin{bmatrix} 1 & -3 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$$

Exercise 6. Show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

(i) $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

ANS: Write $T(\vec{x})$ and $\vec{x}$ as column vectors. Since $\vec{x}$ has 2 entries, A has 2 columns. Since $T(\vec{x})$ has 4 entries, A has 4 rows.

$$\begin{bmatrix} 2x_2 - 3x_1 \\ x_1 - 4x_2 \\ 0 \\ x_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} 3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(ii) $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

ANS: Similar to part (i), we have

$$\begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

TABLE 1 Reflections

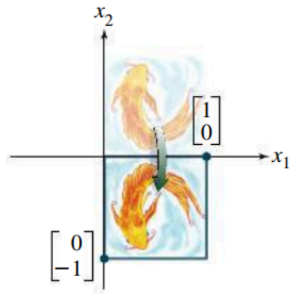
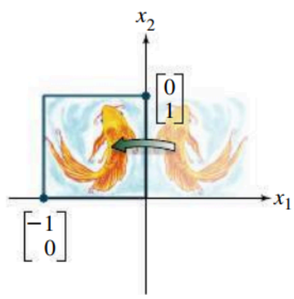
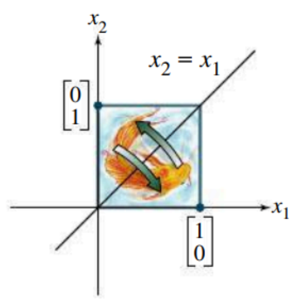
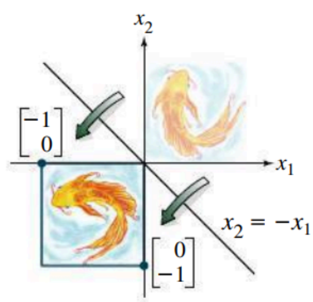
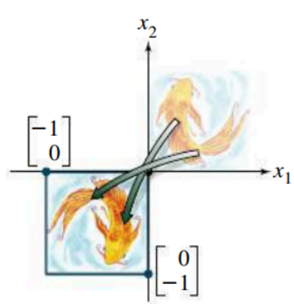
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

TABLE 2 **Contractions and Expansions**

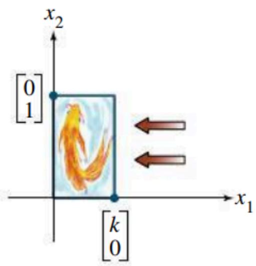
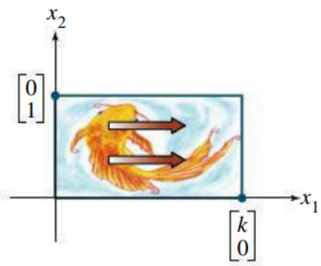
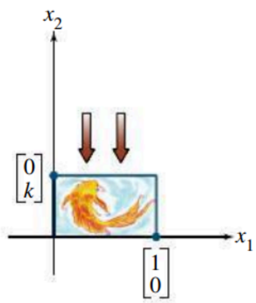
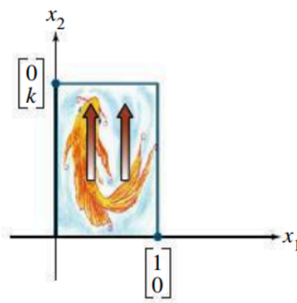
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

TABLE 3 **Shears**

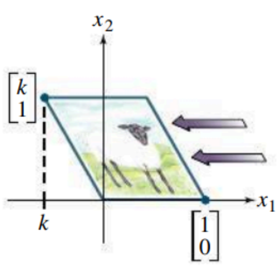
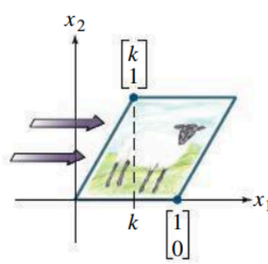
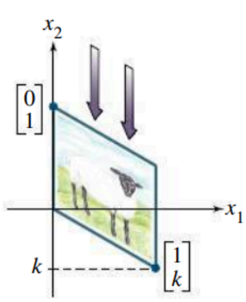
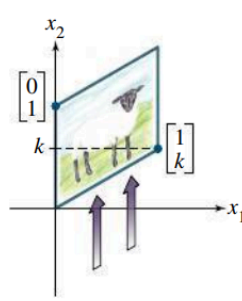
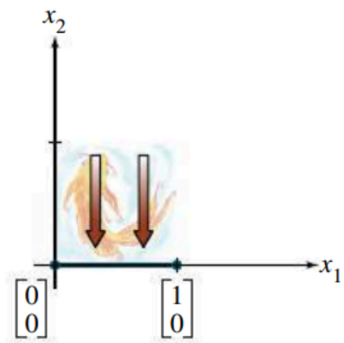
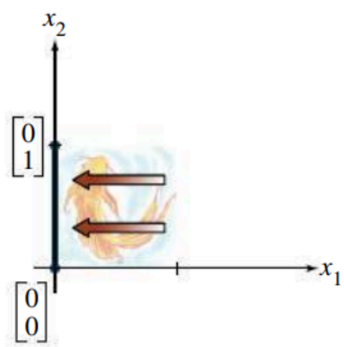
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$